# Order matters: Boosting Mathematical Solver via Machine Learning Techniques

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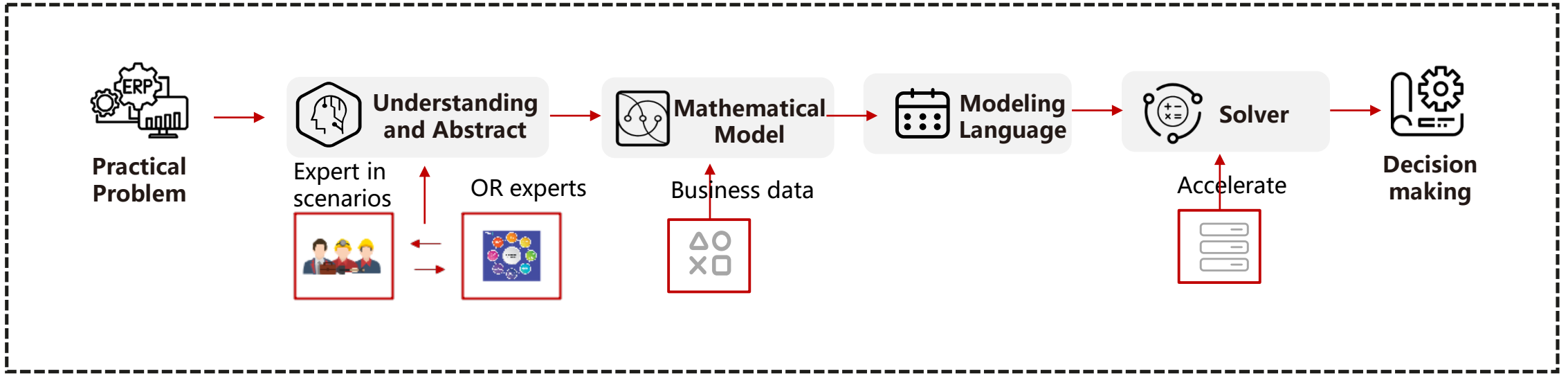
# Outline

- Preliminary
- Order Matters: Why We Cares
- Two of Our Group’s Research Work in Year 2022
  - > Accelerating Linear Programming Solving by Exploiting the Performance Variability via Reinforcement Learning
  - > Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model
- Challenges and Outlook

# Preliminary




**The procedure of calling solver to solve mathematical optimization problem**

# Preliminary

## Mixed-Integer Linear Program

$$\begin{aligned} \arg\min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}, \\ & \mathbf{x} \in \mathbb{Z}^p \times \mathbb{R}^{n-p}. \end{aligned}$$

- $\mathbf{c} \in \mathbb{R}^n$ the objective coefficients
- $\mathbf{A} \in \mathbb{R}^{m \times n}$ the constraint coefficient matrix
- $\mathbf{b} \in \mathbb{R}^m$ the constraint right-hand-sides
- $\mathbf{l}, \mathbf{u} \in \mathbb{R}^n$ the lower and upper variable bounds
- $p \leq n$ integer variables

## Linear Program (LP) relaxation

$$\begin{aligned} \arg\min_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{l} \leq \mathbf{x} \leq \mathbf{u}, \\ & \mathbf{x} \in \mathbb{R}^n. \end{aligned}$$

Convex problem, efficient algorithms (e.g., simplex).

- $\mathbf{x}^\star \in \mathbb{Z}^p \times \mathbb{R}^{n-p}$ (lucky) $\rightarrow$ solution to the original MILP
- $\mathbf{x}^\star \notin \mathbb{Z}^p \times \mathbb{R}^{n-p} \rightarrow$ lower bound to the original MILP

## Branch and Bound

Stopping criterion:

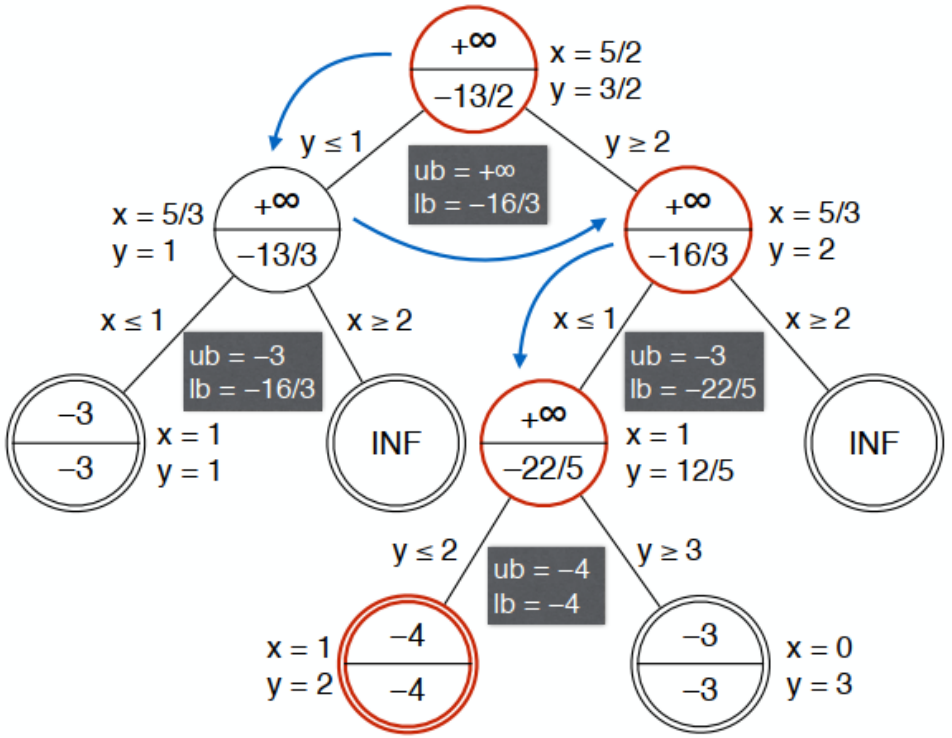
- L = U (optimality certificate)
- L = $\infty$ (infeasibility certificate)
- L - U < threshold (early stopping)

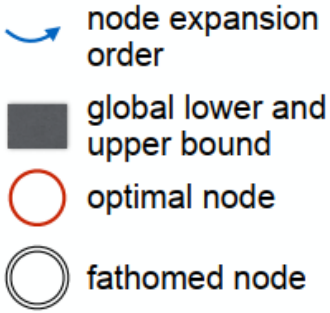
Split the LP recursively over a non-integral variable, i.e. $\exists i \leq p \mid x_i^\star \notin \mathbb{Z}$

$$x_i \leq \lfloor x_i^\star \rfloor \quad \vee \quad x_i \geq \lceil x_i^\star \rceil.$$

Lower bound (L): minimal among leaf nodes.
Upper bound (U): minimal among integral leaf nodes.

# Preliminary



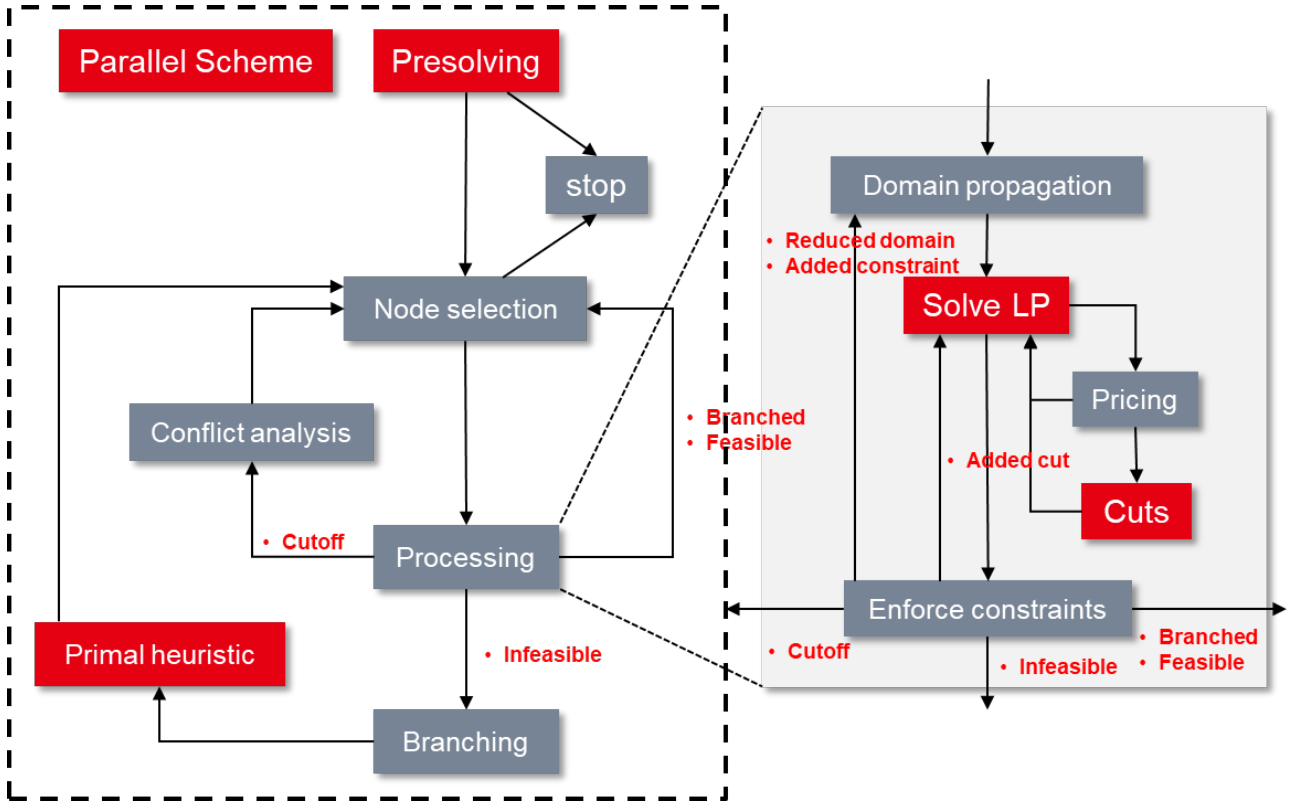




- LP relaxations
- Upper and lower bound
- Branching

$$\begin{aligned} \min\ & -2x - y \\ \text{s.t.}\ & 3x - 5y \le 0 \\ & 3x + 5y \le 15 \\ & x \ge 0,\ y \ge 0 \\ & x, y \in Z \end{aligned}$$

**The illustration of branch-and-bound framework**

# Preliminary




**Presolve:** reduces the dimension of the input problems
**Node selection:** determines which node to solve in the B&B tree
**Branching:** determines which fractional variable to solve in the B&B tree
**Cut separation:** generates corresponding constraints to shrink the search space
**Primal heuristics:** aims to find the primal feasible solution ASAP
**Solve LP:** calls primal and dual simplex to solve LP relaxation
**Domain propagation:** deduces the narrowed bound for variables and propagates these bounds

**Solving procedure of MILP solver**

# Order matters: why we care

Prof. Andrea Lodi [1]

1. "The performance of MIP solvers is subject to some unexpected variability that appears, for example, when changing from one computing platform to another, when permuting rows and/or columns of a model, etc."
2. "This phenomenon has certainly been observed for decades and has been the topic of an extensive literature in the artificial intelligence and SATisfiability communities."
3. "One source of performance variability is rooted in the so-called imperfect tie breaking. Most of the decisions taken by an MIP solver are based on ordering candidates according to scores and selecting the candidate with the best score. This is true for cut separation and cut filtering as well as for one of the most crucial decisions of the MIP solution process, namely, the variable to branch on at each node."

Dr. Ivet Galabova [2], Developer of HiGHS

For some problems the presolve performance is very sensitive to the order in which the rules are applied. Hence, an analysis of presolve would be incomplete without an investigation of this effect for particular LP instances. Alternative orderings may be more suitable for a problem, depending on the problem structure.

[1] Lodi, Andrea, and Andrea Tramontani. "Performance variability in mixed-integer programming." Theory driven by influential applications. INFORMS, 2013. 1-12.
[2] Galabova, Ivet. "Presolve, crash and software engineering for HiGHS." (2023).

# Accelerating Linear Programming Solving by Exploiting the Performance Variability via Reinforcement Learning[1]

## Introduction

**An easily overlooked phenomenon:** the appearing order of variables in a given LP instance indeed affects the solver's performance. The solver takes input the model constructed by human experts.

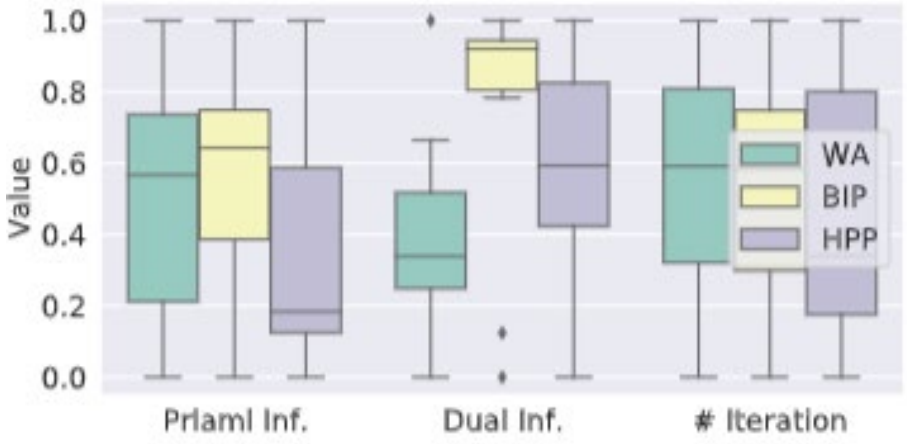



**Figure:** Three distinct LP instances are selected to perform the preliminary experiment. The significant variance of metrics shows that solver performance is quite sensitive to the different formulations for a given LP instance.

| Idx | HPP (m=146722, n=260636, nnz=668270) | | | WA (m= 64282, n=61000, nnz=359428) | | | BIP (m=195, n=1083, nnz=7440) | | |
|---|---|---|---|---|---|---|---|---|---|
| | Primal Inf. | Dual Inf. | # Iteration | Primal Inf. | Dual Inf. | # Iteration | Primal Inf. | Dual Inf. | # Iteration |
| 1 | 7.721295E+09 | 1.884180E+11 | 19408.00 | 311.22 | 1.9980E+09 | 14083.00 | 9.43 | 1.922493E+07 | 539.00 |
| 2 | 7.720309E+09 | 1.882416E+11 | 19242.00 | 295.33 | 1.9790E+09 | 17020.00 | 9.58 | 1.937180E+07 | 486.00 |
| 3 | 7.725508E+09 | 1.869514E+11 | 19208.00 | 269.29 | 1.9530E+09 | 16495.00 | 9.38 | 1.916960E+07 | 557.00 |
| 4 | 7.764919E+09 | 1.876644E+11 | 19408.00 | 262.35 | 2.0620E+09 | 15149.00 | 9.47 | 1.932945E+07 | 453.00 |
| 5 | 7.726977E+09 | 1.878485E+11 | 19175.00 | 299.10 | 1.9080E+09 | 16351.00 | 9.37 | 1.915617E+07 | 586.00 |
| 6 | 7.746786E+09 | 1.877944E+11 | 19266.00 | 281.19 | 1.8650E+09 | 14971.00 | 9.43 | 1.922493E+07 | 528.00 |
| 7 | 7.746866E+09 | 1.879217E+11 | 19274.00 | 308.00 | 1.9630E+09 | 15778.00 | 9.22 | 1.706866E+07 | 591.00 |
| 8 | 7.729257E+09 | 1.873669E+11 | 19391.00 | 269.86 | 2.1070E+09 | 16586.00 | 9.00 | 1.880115E+07 | 519.00 |
| 9 | 7.745738E+09 | 1.875424E+11 | 19203.00 | 296.18 | 2.0280E+09 | 14984.00 | 9.22 | 1.902808E+07 | 453.00 |
| 10 | 7.727762E+09 | 1.882426E+11 | 19240.00 | 284.81 | 2.2290E+09 | 15861.00 | 9.05 | 1.674231E+07 | 554.00 |

**Motivation:** using machine learning technique to automatically find better formulation for solvers.

**Three challenges to introduce ML techniques:**
1) How to appropriately represent an LP instance
2) What machine learning model is suitable
3) How to efficiently train above inferring model

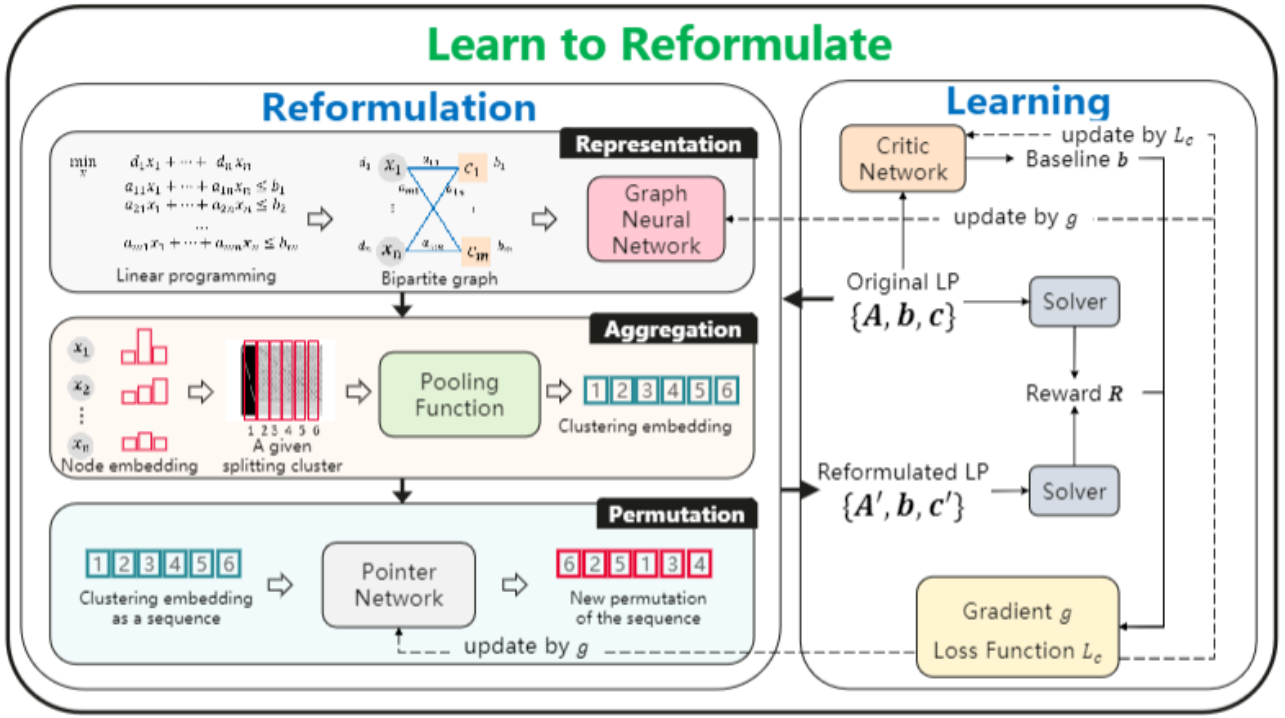
[1] Xijun Li, Qingyu Qu, Fangzhou Zhu, Jia Zeng, Mingxuan Yuan, Jie Wang: Accelerating Linear Programming Solving by Exploiting the Performance Variability via Reinforcement Learning. AAAI 2023 workshop on AI to Accelerate Science and Engineering

* This method has been filed a patent for Huawei.

# Accelerating Linear Programming Solving by Exploiting the Performance Variability via Reinforcement Learning

## Proposed method: overview

- **Representation:** The inputting LP instance is represented by a bipartite graph, and then the embedding of variables is obtained via a GNN
- **Aggregation:** The embedding of variables will be further aggregated with a given group of variable clusters
- **Permutation:** Taking as input the previous embeddings, a pointer network (PN) is used to output a new permutation of variables
- **Learning:** The learning part interacts with the reformulation part to update the parameters of GNN and PN, trained with REINFORCE algorithm.



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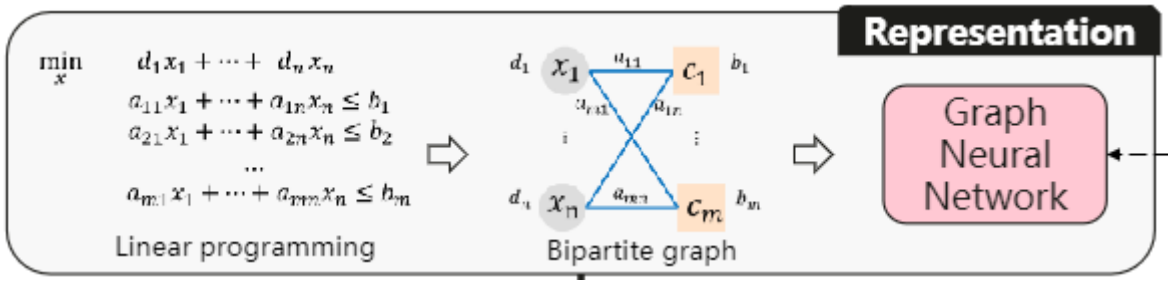
# Accelerating Linear Programming Solving by Exploiting the Performance Variability via Reinforcement Learning

## Proposed method: reformulation part




- **Representation:** The inputting LP instance is represented by a bipartite graph and get representation via a graph convolutional neural network (GCNN).

We adopt the same method as done in (Gasse et al. 2019) to represent a given linear programming as a bipartite graph $(\mathcal{G}, \mathbf{C}, \mathbf{E}, \mathbf{V})$. Specifically, in the bipartite graph, $\mathbf{C} \in \mathbb{R}^{m \times c}$ corresponds to the features of the constraints in the LP; $\mathbf{V} \in \mathbb{R}^{n \times d}$ denotes the features of the variables in the LP; and an edge $e_{ij} \in \mathbf{E}$ between a constraint node $i$ and a variable node $j$ if the corresponding coefficient $\mathbf{A}_{i,j} \neq 0$. For simplicity, we just attach the value of $\mathbf{A}_{i,j}$ to the corresponding edge $e_{ij}$. Readers can refer to the used features in Appendix.

One layer representation of constraint

$$\mathbf{c}_i^{(k+1)} \leftarrow \mathbf{f}_{\mathbf{C}} \left( \mathbf{c}_i^{(k)}, \sum_{j}^{(i,j)\in\mathbf{E}} \mathbf{g}_{\mathbf{C}}(\mathbf{c}_i^{(k)}, \mathbf{v}_j^{(k)}, e_{ij}) \right),$$

$$\mathbf{v}_j^{(k+1)} \leftarrow \mathbf{f}_{\mathbf{V}} \left( \mathbf{v}_j^{(k)}, \sum_{i}^{(i,j)\in\mathbf{E}} \mathbf{g}_{\mathbf{V}}(\mathbf{c}_i^{(k)}, \mathbf{v}_j^{(k)}, e_{ij}) \right)$$

One layer representation of variable



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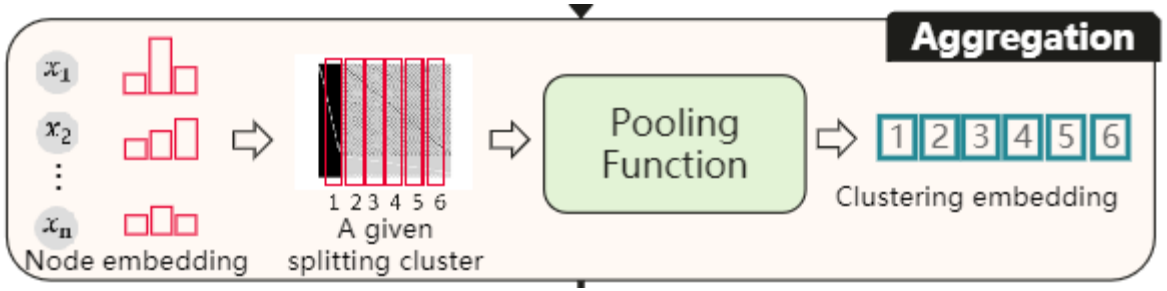
# Accelerating Linear Programming Solving by Exploiting the Performance Variability via Reinforcement Learning

## Proposed method: reformulation part




- **Aggregation:** The embedding of variables are aggregated with a given group of variable/constraint clusters **due to the combinatorial explosion of search space**

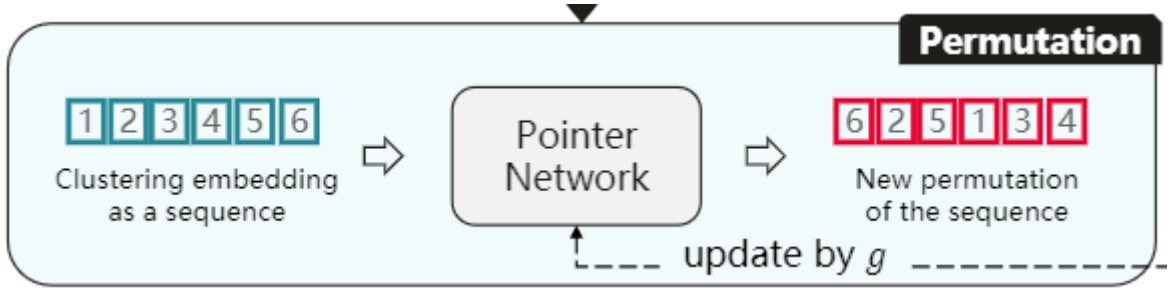
  - **Splitting:** For a given LP instance, the variables/constraints are split up into many disjoint cluster. *Note that variables within one cluster are subject to the order of variables in the original LP.* The clustering method is not restricted. *It could be specified by human experts or using the hyper-graph decomposition method*
  - **Pooling function:** With above splitting clusters and variable embedding, we perform the aggregation for each cluster via pooling function

$$\Sigma_j = \mathcal{P}(\{\mathbf{v}_i | x_i \in C_j\})$$

where the pooling function that could be maximum, minimum, average, or other appropriate functions. We still do not restrict the kind of pooling function here.

# Accelerating Linear Programming Solving by Exploiting the Performance Variability via Reinforcement Learning

## Proposed method: reformulation part




- **Permutation:** Taking as input the previous embeddings, a pointer network (PN) is used to output a new permutation of variables
  - Given an LP, we reformulate it by reordering its variables/constraints
  - More specifically, we want to learn a policy that can output a better formulation for a given LP

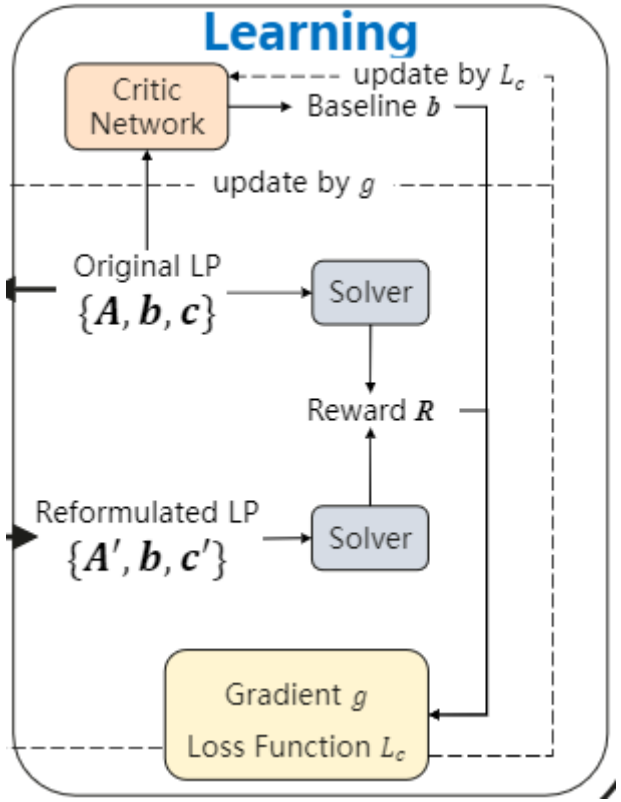
**Reformulation policy** $p(\pi|\{C_j\}_{j=1}^M) = \prod_{j=1}^{M} p(\pi(j)|\pi(<j), \{C_j\}_{j=1}^M)$

a splitting clustering

the permutation of given splitting clustering

# Accelerating Linear Programming Solving by Exploiting the Performance Variability via Reinforcement Learning

## Proposed method: learning part




**Policy**

$$p(\pi|\{C_j\}_{j=1}^M) = \prod_{j=1}^{M} p(\pi(j)|\pi(<j), \{C_j\}_{j=1}^M)$$

- $\{C_j\}_{j=1}^M$: a splitting clustering
- $\pi$: the permutation of given splitting clustering

**Reward**

$$R(\pi|l) = 1 - \frac{\mathcal{S}_{\mathcal{M}}(l|\pi)}{\mathcal{S}_{\mathcal{M}}(l)}$$

- $\mathcal{S}_{\mathcal{M}}(l|\pi)$: the solving performance of the reformulated LP instance
- $l$: a given LP instance
- $\mathcal{S}_{\mathcal{M}}(l)$: the solving performance of the input LP instance

**Optimization**

$$J(\theta_G, \theta_P|l) = \mathbb{E}_{\pi \sim p_{\theta_G, \theta_P}(.|l)}[R(\pi|l)]$$

- $\theta_G$: the parameterized representation (i.e., GCN)
- $\theta_P$: the parameterized policy (i.e., pointer network)



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# Accelerating Linear Programming Solving by Exploiting the Performance Variability via Reinforcement Learning

**Experimental evaluation: benchmark**

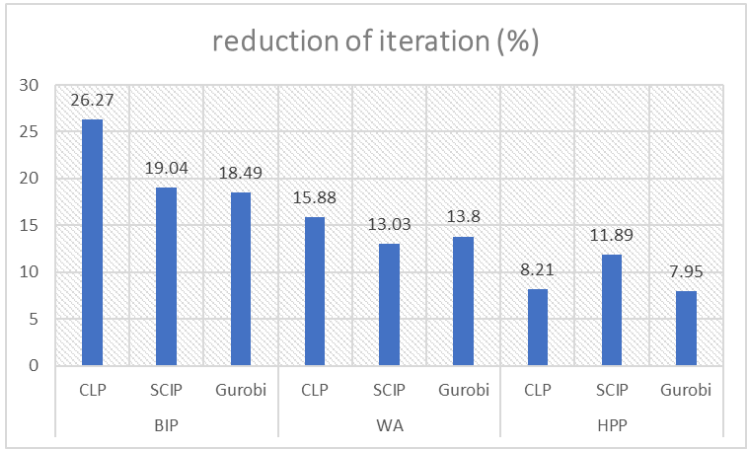
- **Datasets:** three sets of Mixed Integer Linear Programming (MILP) problems. All datasets are scenario specific.
  - Two of them, Balanced Item Placement (BIP) and Workload Apportionment (WA), are from NeurIPS 2021 ML4CO competition
  - The third one is obtained from a real-life production planning scenario called HPP.
- **Testing solver:** Gurobi (commercial), SCIP and CLP (open-source)
- **Metrics:** we measure the improvement gained from reformulation
  - Solving time
  - Iteration number of the search process

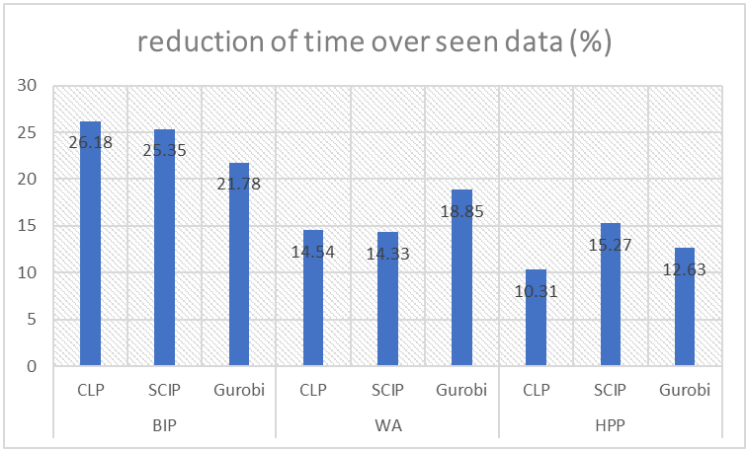
Table 1: Statistical description of used dataset

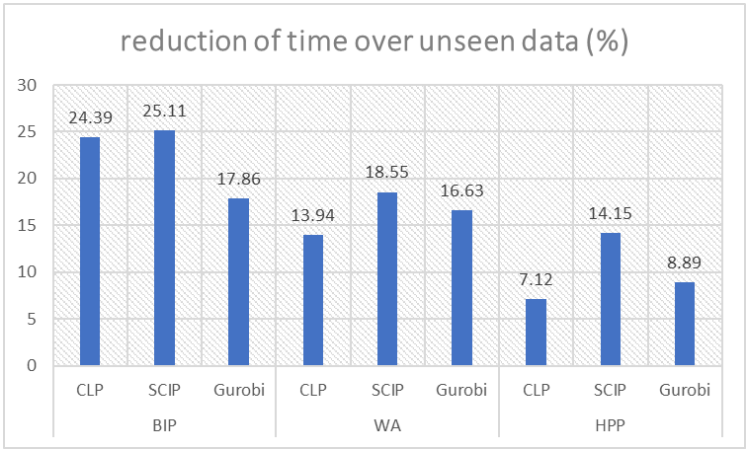
| Dataset | $m$ | $n$ | $NNZ$ |
|---|---|---|---|
| BIP | $195.00 \pm 00.00$ | $1083.00 \pm 00.00$ | $7440.00 \pm 00.00$ |
| WA | $6.43e04 \pm 54.51$ | $6.1e04 \pm 00.00$ | $3.62e05 \pm 6007.41$ |
| HPP | $1.25e06 \pm 6.93e04$ | $2.66e06 \pm 1.83e05$ | $6.64e06 \pm 4.29e05$ |

# Accelerating Linear Programming Solving by Exploiting the Performance Variability via Reinforcement Learning

## Experimental evaluation: numerical results




**Take-away messages:**

1) Various solvers (including commercial and open-source) can benefits from reformulation.
2) Gurobi is the most robust to the varying formulation.



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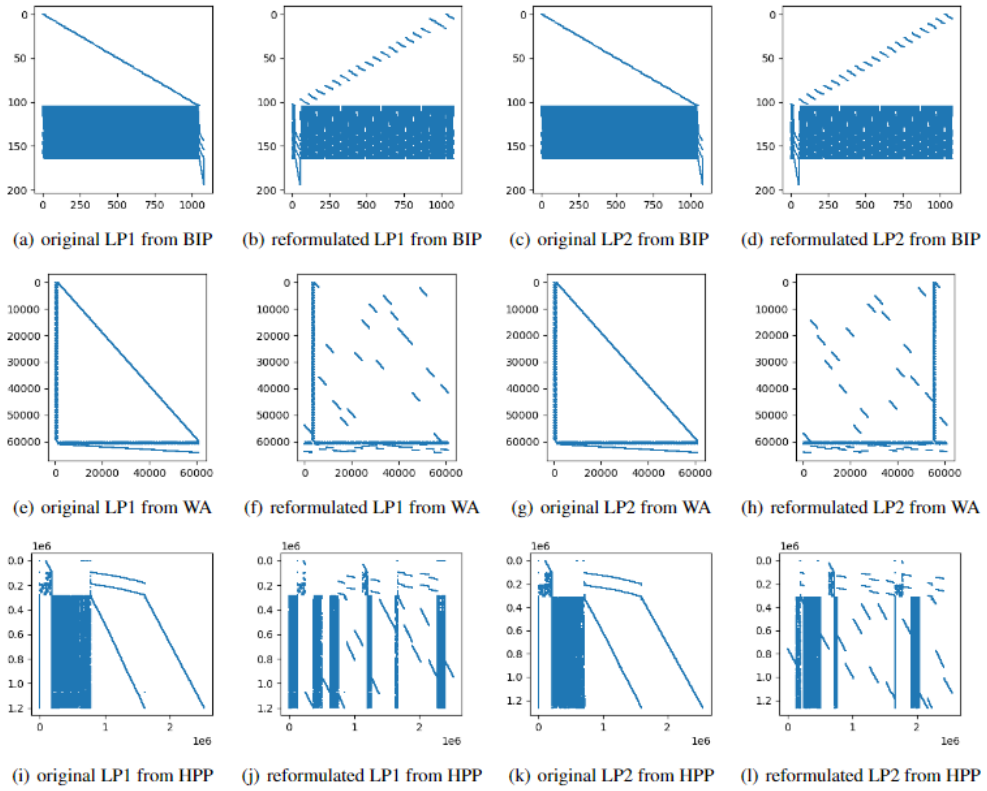
# Accelerating Linear Programming Solving by Exploiting the Performance Variability via Reinforcement Learning

## Hindsight: what better formulation the agent found



(a) original LP1 from BIP (b) reformulated LP1 from BIP (c) original LP2 from BIP (d) reformulated LP2 from BIP

(e) original LP1 from WA (f) reformulated LP1 from WA (g) original LP2 from WA (h) reformulated LP2 from WA

(i) original LP1 from HPP (j) reformulated LP1 from HPP (k) original LP2 from HPP (l) reformulated LP2 from HPP

Figure 5: Visualization of reformulation process over BIP, WA and HPP datasets

1) **Efficacy:** our proposed reformulation method captures the characteristics of LP instances originating from different scenarios.

2) **Different captured patterns on various dataset:** The reformulation is relatively stable when the original LP instances are highly similar. All the original LP instances of BIP have the same number of constraints and variables, which is only different in the value of coefficients. because the corresponding original LP instances differ in not only the value of coefficients but also the number of constraints and variables.

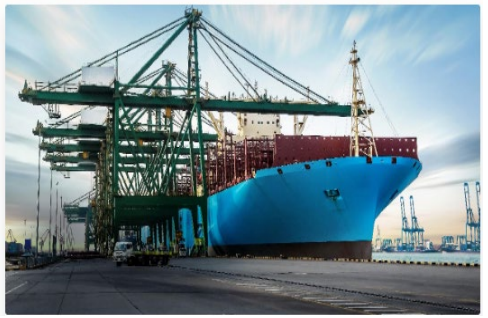
# Accelerating Linear Programming Solving by Exploiting the Performance Variability via Reinforcement Learning

## Summary of this work

- In this paper, we propose a reinforcement learning-based reformulation method to accelerate the linear programming solving. To the best of our knowledge, this is the first work that exploits the performance variability of modern solvers via machine learning techniques to gain performance.
- The idea of using machine learning techniques to exploit or to reduce the performance variability of mathematical programming solvers can be extended in many directions.
  - One can further **learn the modeling experience or gain the modeling tricks** from the reformulation derived from the neural networks.
  - One can use the proposed method to decide the **better ordering rules in various decision-making components in solver such as pricing, variable selection, cut separation**, etc.
  - We believe that this work can inspire the future research to better exploit the performance variability of solvers to improve the solvers.

# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

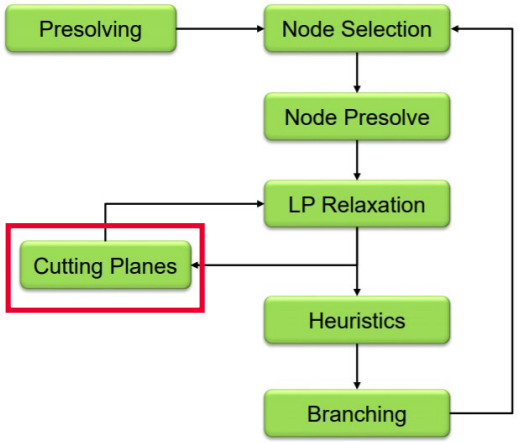
Production Planning



Port Dispatching

…

- Mixed Integer Programming **formulates** a wide range of **important real-world applications**




Pipeline of solving MILPs

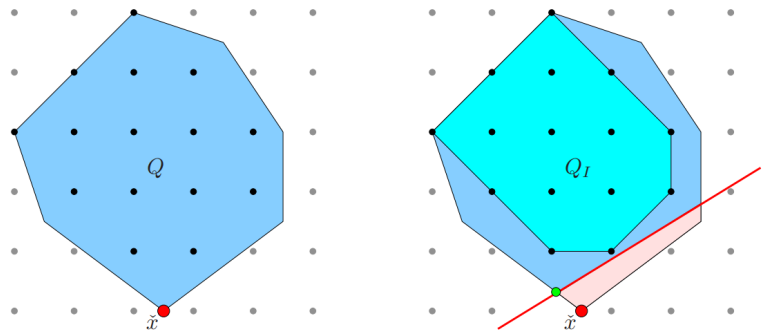



**Figure 2.3.** A cutting plane that separates the fractional LP solution $\tilde{x}$ from the convex hull $Q_I$ of integer points of $Q$.

Example of cutting planes

- **Cutting planes** are important for improving the efficiency of solving MILPs
- **Cut selection** aims to select a proper subset of candidate cuts

[1] Xijun Li*, Zhihai Wang*, Jie Wang, Yufei Kuang, Mingxuan Yuan, Jia Zeng, Yongdong Zhang, Feng Wu: Learning Cut Selection for Mixed-Integer Linear Programming via Hierarchical Sequence Model. ICLR 2023

* This method has been filed a patent for Huawei.

# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

## Introduction: cut selection

- While many classes of cutting-planes are at the disposal of integer programming solvers, our scientific understanding is far from complete with regards to cutting-plane selection, i.e., **the task of selecting a portfolio of cutting-planes to be added to the LP relaxation** at a given node of the branch-and-bound tree [1].
- The large number of cutting planes generated by these separators, however, can pose a computational problem. Therefore, **a sophisticated cut management is indispensable** [2].
- Adding all of these cuts to the LP does not yield the best overall performance. Therefore**, a selection criterion is needed in order to identify a "good" subset of the generated cuts** [3].

Separator Stats :

| Separators | Time | Calls | VioCuts | UniCuts | Applied | DelPool | DelLP | AvgVio | AvgLPVio |
|---|---|---|---|---|---|---|---|---|---|
| cutpool | 5.16 | 0 | 28488 | 20480 | 2529 | 16028 | 1638 | 0.454045 | 0.119761 |
| IMPL_0 | 0.22 | 64 | 1221 | 866 | 335 | 679 | 214 | 0.086314 | 0.109228 |
| Parity_0 | 0.21 | 64 | 83 | 49 | 21 | 33 | 10 | 0.146949 | 0.149105 |
| BinPack_0 | 0.01 | 64 | 13 | 10 | 11 | 2 | 3 | 0.114480 | 0.133982 |
| DJ_0 | 0.28 | 64 | 114 | 72 | 27 | 55 | 17 | 0.101793 | 0.156583 |
| PATH_0 | 7.19 | 64 | 13409 | 9816 | 1008 | 8247 | 639 | 0.645638 | 0.143585 |
| ModK_0 | 0.74 | 50 | 744 | 543 | 181 | 419 | 112 | 0.181118 | 0.250318 |
| AGGR_0 | 4.36 | 43 | 3 | 3 | 3 | 1 | 1 | 4.202781 | 0.745435 |
| TwoRowDJ_0 | 2.23 | 50 | 154 | 130 | 10 | 123 | 4 | 0.161719 | 0.170186 |
| GMI_0 | 7.36 | 50 | 10595 | 6878 | 179 | 6469 | 117 | 0.342388 | 6.643384 |
| Presolve_0 | 0.00 | 64 | 2152 | 2113 | 754 | 0 | 521 | 0.115196 | 0.144838 |

Snapshot of OptVerse Solver 's Log

**The task of selecting a portfolio of cutting-planes is a NP-hard problem !**

**Cut selection: trade-off between tightening LP relaxations and computational burden of cuts**


[1] Wesselmann, Franz, and U. Stuhl. "Implementing cutting plane management and selection techniques." Technical Report. University of Paderborn, 2012.
[2] Dey, Santanu S., and Marco Molinaro. "Theoretical challenges towards cutting-plane selection." Mathematical Programming 170 (2018): 237-266.
[3] Achterberg, Tobias. Constraint integer programming. Diss. 2007.

# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

## Limitations of existing methods: heuristics

Prof. Yoshua Bengio [1]

Prof. Andrea Lodi [1]

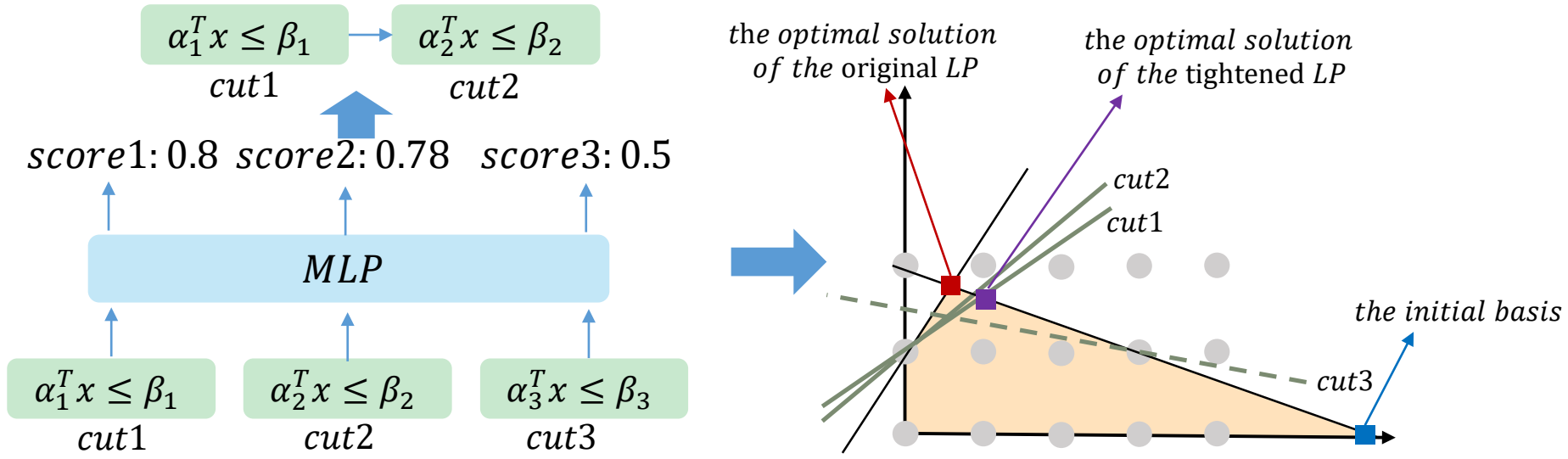
1. However, assuming that MILPs come from a specific distribution, there has been **a recent surge in research related to statistical approaches to learning such heuristics**.
2. From the CO point of view, **machine learning can help improve an algorithm** on a distribution of problem instances in two ways. On the one side, …… On the other side, ……
3. Indeed, learning mechanisms able to **discover structural properties of seemingly equivalent components** of an algorithm would certainly be very useful

**Machine learning offers a promising approach to learn more effective heuristics from MILPs collected from specific applications**

[1] Bengio, Yoshua, Andrea Lodi, and Antoine Prouvost. "Machine learning for combinatorial optimization: a methodological tour d' horizon." *European Journal of Operational Research* 290.2 (2021): 405-421.

# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

## Limitations of existing methods: learning-based

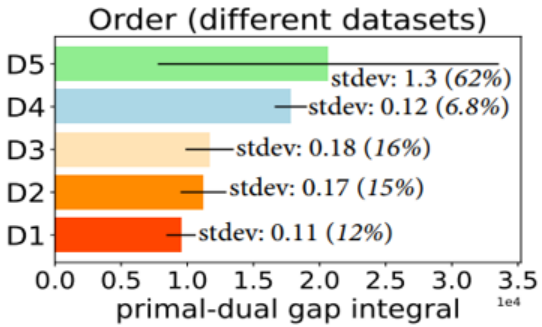



**Ilustration of Score-based policy**

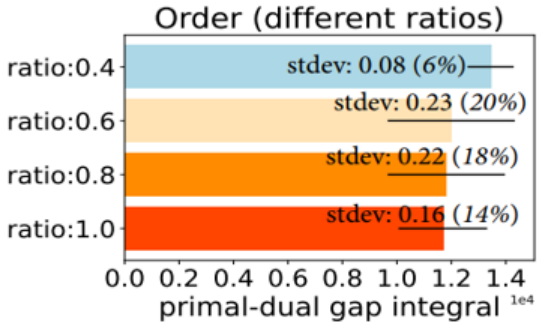
- Neglecting learning **how many cuts to select**
- **Order of selected cuts** is important for the solving efficiency
- Do not take into account **the interaction among cuts**

# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

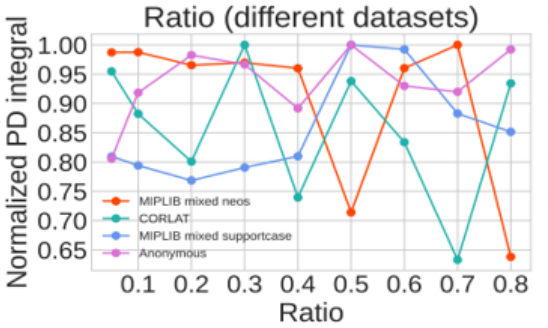
## Three key problems in cut selection



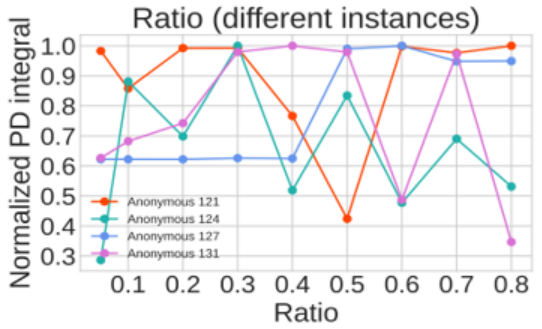

(a) Evaluate RandomAll on five different datasets.




(b) Evaluate RandomNv on MIPLIB mixed neos.




(c) Evaluate NV with different ratios on four datasets.




(d) Evaluate NV with different ratios on Anonymous.

(**P1**) Which cuts should be preferred (**P2**) How many cuts to select (**P3**) What **order** to prefer (**we first observe**)

**Order and ratio of selected cuts significantly impact the efficiency of solving MILPs**



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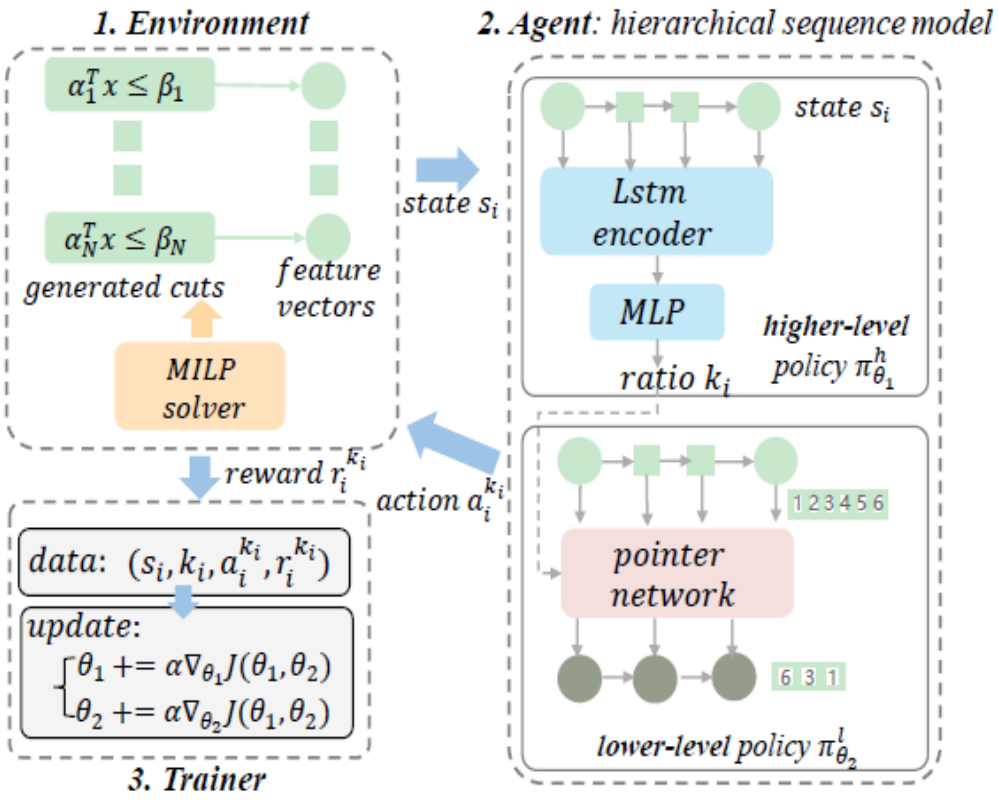
# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

## Proposed method: overview

- **Reinforcement Learning Formulation**
  - Formulate MILP solver as the environment
  - Formulate cut selection policy as the agent
  - Time/Primal-dual gap integral as reward
- **Hierarchical Sequence Model**
  - We **first** formulate cut selection as a sequence to sequence learning problem
  - Propose Hierarchical Sequence Model to model the agent
- **Training Algorithm**
  - We derive a hierarchical policy gradient



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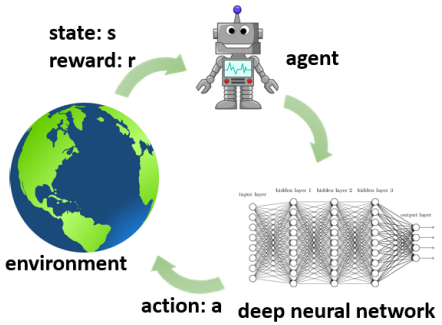
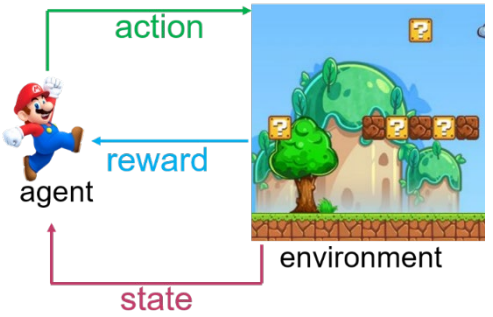
# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

## Proposed method: *Markov* decision process formulation

- State space $S$: We define each state by **a sequence** of thirteen-dimensional feature vectors (see Appendix F.1).
- Action space $A$: All the **ordered subsets** of the candidate cuts
- Reward function $r$: **Solving time**/**Primal-dual gap integral**/Dual bound improvement
- Transition function $f$: A **deterministic** mapping from the current state and action to the next state
- The terminal state. There is no standard and unified criterion to determine when to terminate the cut separation procedure.
  - The number of cut separation rounds is a **hyperparameter** in MILP solvers.
  - We set the cut separation rounds as one and add cuts at the root node in this paper.
  - As a result, the formulation is a **contextual bandit**.



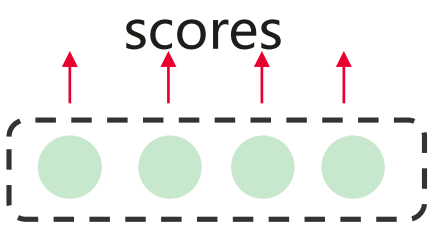



# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

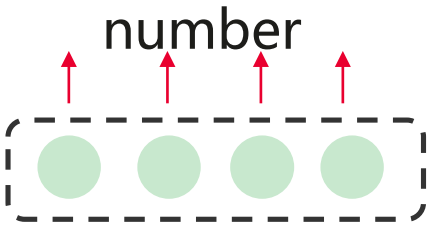
**Proposed method: cut selection policy**

- Cut selection policy: $\pi: S \to P(A)$, where $P(A)$ denotes the **probability distribution** over the action space
  - Tackle (P1)-(P3) simultaneously
  - Exploration and differentiability

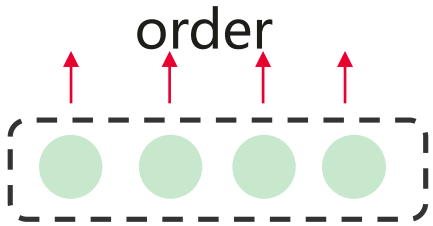
- Directly learning such policies is challenging
  - the **cardinality** of the action space can be extremely **large** due to its combinatorial structure
  - the **length** and max length of actions are **variable** across different MILPs




**(P1) Which cuts to prefer**



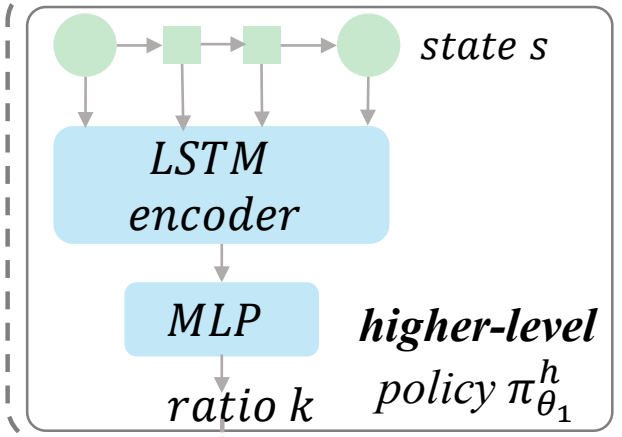

**(P2) How many cuts to select**



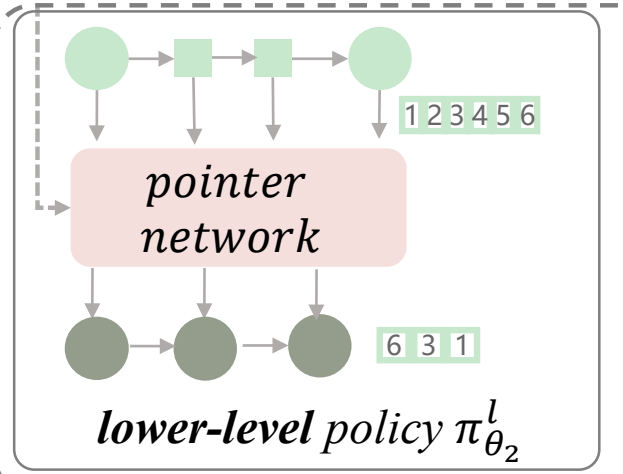

**(P3) What order of selected cuts to prefer (Our observation)**

# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

## Proposed method: hierarchical sequence model




- The higher-level policy $\pi^h: S \to P([0,1])$
  - Model the higher-level policy as a **tanh-Gaussian**
  - An LSTM encoder to encode the input cuts
  - A MLP to predict the mean and variance of the Gaussian




- The lower-level policy $\pi^l: S \times [0,1] \to P(A)$
  - Model the lower-level policy as a **Sequence to Sequence model**
  - The number of candidate cuts **varies** on different instances
  - A **pointer network** to select an ordered subset

# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

## Proposed method: hierarchical policy gradient

$$J(\theta) = \mathbb{E}_{s\sim\mu, a_k\sim\pi_\theta(\cdot|s)}[r(s, a_k)],$$

Optimization objective

**Proposition 1.** *Given the cut selection policy* $\pi_\theta(a_k|s) = \mathbb{E}_{k\sim\pi^h_{\theta_1}(\cdot|s)}[\pi^l_{\theta_2}(a_k|s,k)]$ *and the training objective (3), the hierarchical policy gradient takes the form of*

$$\nabla_{\theta_1} J([\theta_1,\theta_2]) = \mathbb{E}_{s\sim\mu,k\sim\pi^h_{\theta_1}(\cdot|s)}[\nabla_{\theta_1}\log(\pi^h_{\theta_1}(k|s))\mathbb{E}_{a_k\sim\pi^l_{\theta 2}(\cdot|s,k)}[r(s,a_k)]],$$

$$\nabla_{\theta_2} J([\theta_1,\theta_2]) = \mathbb{E}_{s\sim\mu,k\sim\pi^h_{\theta_1}(\cdot|s),a_k\sim\pi^l_{\theta_2}(\cdot|s,k)}[\nabla_{\theta_2}\log\pi^l_{\theta_2}(a_k|s,k)r(s,a_k)].$$

Hierarchical policy gradient

- HEM leverages the hierarchical structure of the cut selection task, which is important for **efficient exploration** in complex decision-making tasks
- We train HEM via gradient-based algorithms, which is **sample efficient**

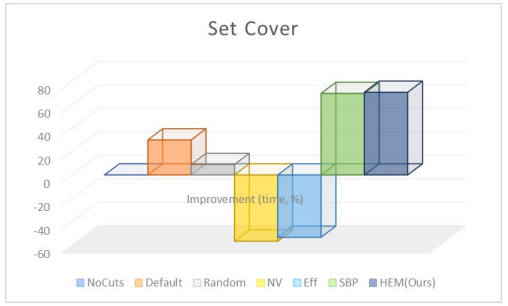
# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

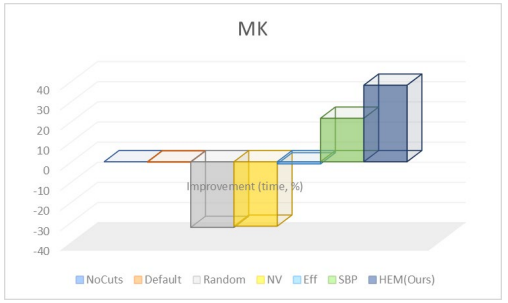
**Experimental evaluation: benchmark**

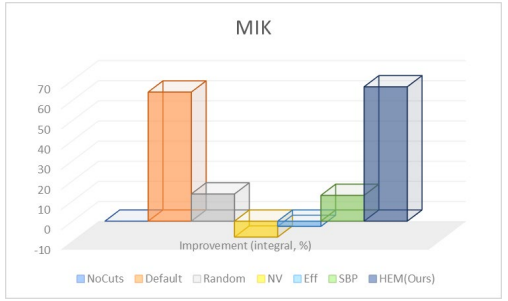
- Datasets: **nine** MILP benchmarks
  - **Easy**: Set Covering, Maximum Independent Set, Knapsack (synthetic)
  - **Medium**: MIK (knapsack constraints), CORLAT (a real dataset used for the construction of a wildlife corridor)
  - **Hard**: Load balancing (**Google**), Anonymous (a large-scale industrial application), **MIPLIB 2017**
- Baselines
  - **five** widely used human-designed cut selection rules
    - NoCuts, Random, Normalized Violation (NV), Efficacy (Eff), and Default
  - a state-of-the-art (SOTA) learning-based method, score-based policy (SBP)
- Metrics
  - Time: solving time
  - PD integral: **Primal-dual gap integral**

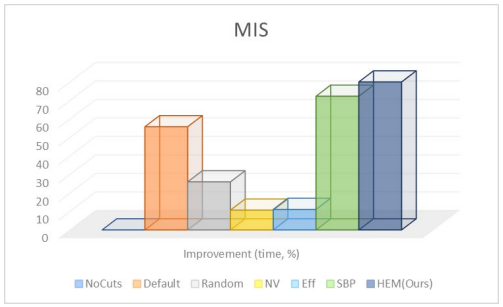
# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

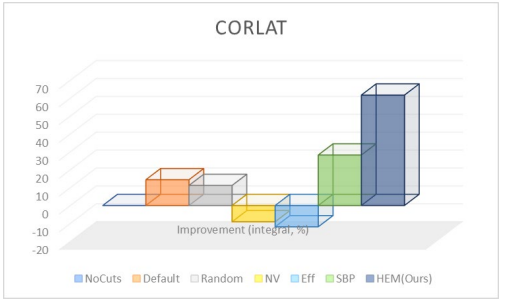
## Experimental evaluation: significantly improves the efficiency

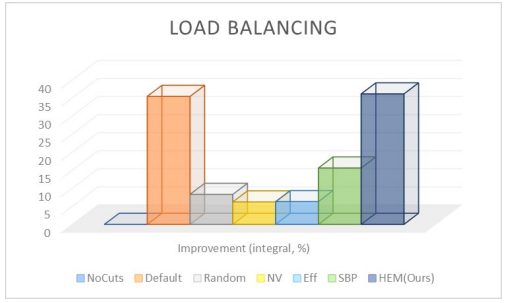


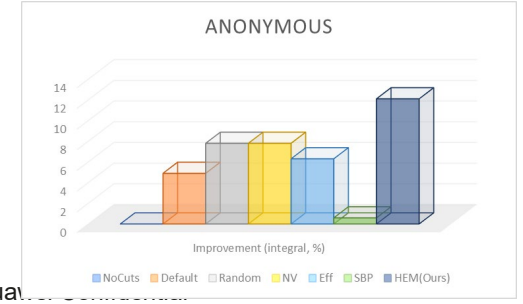



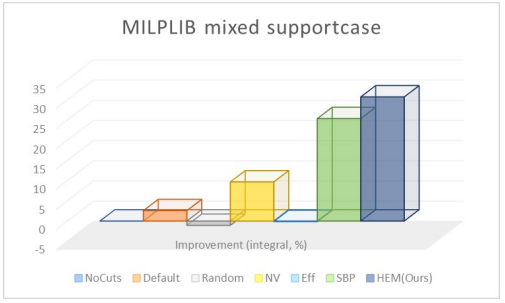



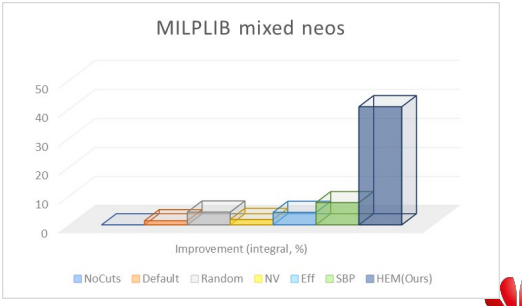



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# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

## Experimental evaluation: ablation studies

Table 2: Comparison between HEM and HEM without the higher-level model.

| | Easy: Maximum Independent Set ($n = 500,\ m = 1953$) | | | Medium: Corlat ($n = 466,\ m = 486$) | | | Hard: MIPLIB mixed neos ($n = 6958,\ m = 5660$) | | |
|---|---|---|---|---|---|---|---|---|---|
| Method | Time(s) ↓ | Improvement ↑ (Time, %) | PD integral ↓ | Time(s) ↓ | PD integral ↓ | Improvement ↑ (PD integral, %) | Time(s) ↓ | PD integral ↓ | Improvement ↑ (PD integral, %) |
| NoCuts | 8.78 (6.66) | NA | 71.31 (51.74) | 103.30 (128.14) | 2818.40 (5908.31) | NA | 253.65 (80.29) | 14652.29 (12523.37) | NA |
| Default | 3.88 (5.04) | 55.81 | 29.44 (35.27) | 75.20 (120.30) | 2412.09 (5892.88) | 14.42 | 256.58 (76.05) | 14444.05 (12347.09) | 1.42 |
| SBP | 2.43 (5.55) | 72.32 | 21.99 (40.86) | 70.41 (122.17) | 2023.87 (5085.96) | 28.19 | 256.48 (78.59) | 13531.00 (12898.22) | 7.65 |
| HEM w/o H | 1.88 (4.20) | 78.59 | 16.70 (28.15) | 63.14 (115.26) | 1939.08 (5484.83) | 31.20 | 249.21 (88.09) | 13614.29 (12914.76) | 7.08 |
| HEM (Ours) | **1.76 (3.69)** | **79.95** | **16.01 (26.21)** | **58.31 (110.51)** | **1079.99 (2653.14)** | **61.68** | **248.66 (89.46)** | **8678.76 (12337.00)** | **40.77** |

Each component of HEM is important for the solving efficiency.

Table 3: Comparison between HEM, HEM-ratio, and HEM-ratio-order.

| | Easy: Maximum Independent Set ($n = 500,\ m = 1953$) | | | Medium: Corlat ($n = 466,\ m = 486$) | | | Hard: MIPLIB mixed neos ($n = 6958,\ m = 5660$) | | |
|---|---|---|---|---|---|---|---|---|---|
| Method | Time(s) ↓ | Improvement ↑ (Time, %) | PD integral ↓ | Time(s) ↓ | PD integral ↓ | Improvement ↑ (PD integral, %) | Time(s) ↓ | PD integral ↓ | Improvement ↑ (PD integral, %) |
| NoCuts | 8.78 (6.66) | NA | 71.31 (51.74) | 103.30 (128.14) | 2818.40 (5908.31) | NA | 253.65 (80.29) | 14652.29 (12523.37) | NA |
| Default | 3.88 (5.04) | 55.81 | 29.44 (35.27) | 75.20 (120.30) | 2412.09 (5892.88) | 14.42 | 256.58 (76.05) | 14444.05 (12347.09) | 1.42 |
| SBP | 2.43 (5.55) | 72.32 | 21.99 (40.86) | 70.41 (122.17) | 2023.87 (5085.96) | 28.19 | 256.48 (78.59) | 13531.00 (12898.22) | 7.65 |
| HEM-ratio-order | 2.30 (5.18) | 73.80 | 21.19 (38.52) | 70.94 (122.93) | 1416.66 (3380.10) | 49.74 | 245.99 (93.67) | 14026.75 (12683.60) | 4.27 |
| HEM-ratio | 2.26 (5.06) | 74.26 | 20.82 (37.81) | 67.27 (117.01) | 1251.60 (2869.87) | 55.59 | **244.87 (95.56)** | 13659.93 (12900.59) | 6.77 |
| HEM (Ours) | **1.76 (3.69)** | **79.95** | **16.01 (26.21)** | **58.31 (110.51)** | **1079.99 (2653.14)** | **61.68** | 248.66 (89.46) | **8678.76 (12337.00)** | **40.77** |

Tackling (P1)-(P3) is important for the solving efficiency

# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

**Experimental evaluation: generalize to instances larger than training data**

TABLE 8: Evaluate the generalization ability of HEM on Set Covering and Maximum Independent Set.

| | Set Covering ($n = 1000$, $m = 1000$, $2\times$) | | | Set Covering ($n = 1000$, $m = 2000$, $4\times$) | | |
|---|---|---|---|---|---|---|
| Method | Time(s) ↓ | Improvement ↑ (time, %) | PD integral ↓ | Time(s) ↓ | Improvement ↑ (time, %) | PD integral ↓ |
| NoCuts | 82.69 (78.27) | NA | 609.43 (524.92) | 284.44 (48.70) | NA | 3215.34 (1019.47) |
| Default | 61.01 (78.12) | 26.22 | 494.63 (545.76) | 149.69 ( 141.92) | 47.37 | 1776.22 (1651.15) |
| Random | 64.44 (73.98) | 22.07 | 520.84 (489.52) | 208.12 (131.52) | 26.53 | 2528.36 (1678.66) |
| NV | 92.05 (80.11) | -11.32 | 725.53 (541.68) | 286.10 (45.47) | -0.58 | 3422.46 (1024.19) |
| Eff | 92.32 (79.33) | -11.64 | 733.72 (538.60) | 286.20 (45.04) | -0.62 | 3437.06 (1043.44) |
| SBP | 3.52 (1.36) | 95.74 | 92.89 (25.83) | 7.62 (6.46) | 97.32 | 256.79 (145.92) |
| HEM (Ours) | **3.33 (0.47)** | **95.97** | **89.24 (14.26)** | **7.40 (5.03)** | **97.40** | **250.83 (131.43)** |

| | Maximum Independent Set ($n = 1000$, $m = 3946$, $4\times$) | | | Maximum Independent Set ($n = 5940$, $m = 1500$, $9\times$) | | |
|---|---|---|---|---|---|---|
| Method | Time(s) ↓ | Improvement ↑ (time, %) | PD integral ↓ | Time(s) ↓ | Improvement ↑ (time, %) | PD integral ↓ |
| NoCuts | 170.06 (100.61) | NA | 874.45 (522.29) | 300.00 (0) | NA | 2019.93 (353.27) |
| Default | 42.40 (76.00) | 48.72 | 198.61 (331.20) | 111.18 (144.13) | 60.91 | 616.46 (798.94) |
| Random | 118.25 (109.05) | -43.00 | 574.33 (516.11) | 245.13 (115.80) | 13.82 | 1562.20 (793.09) |
| NV | 160.30 (101.41) | -93.86 | 784.98 (493.24) | 299.97 (0.49) | -5.46 | 1922.52 (349.67) |
| Eff | 158.75 (100.40) | -91.98 | 779.63 (493.05) | 299.45 (3.77) | -5.28 | 1921.61 (361.26) |
| SBP | 50.55 (89.14) | 38.87 | 253.81 (426.94) | 108.42 (143.68) | 61.88 | 680.41 (903.88) |
| HEM (Ours) | **35.34 (67.91)** | **57.26** | **160.56 (282.03)** | **108.02 (143.02)** | **62.02** | **570.48 (760.65)** |

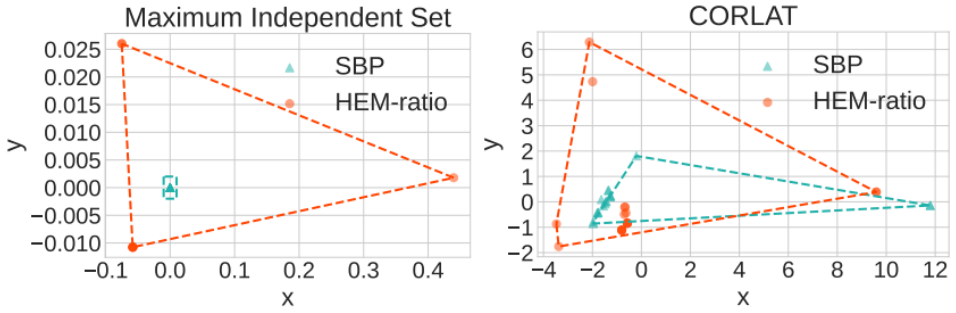
# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

## Evaluation on Huawei production planning and order matching problems

TABLE 9: Evaluation on Huawei production planning and order matching problems.

| | Production planning ($n = 3582.25,\ m = 5040.42$) | | | | Order matching ($n = 67839.68,\ m = 31364.84$) | | | |
|---|---|---|---|---|---|---|---|---|
| Method | Time (s) ↓ | Improvement ↑ (Time, %) | PD integral ↓ | Improvement ↑ (PD integral, %) | Time (s) ↓ | Improvement ↑ (Time, %) | PD integral ↓ | Improvement↑ (PD integral, %) |
| NoCuts | 278.79 (231.02) | NA | 17866.01 (21309.85) | NA | 248.42 (287.29) | NA | 403.41 (345.51) | NA |
| Default | 296.12 (246.25) | -6.22 | 17703.39 (21330.40) | 0.91 | 129.34 (224.24) | 47.93 | 395.80 (341.23) | 1.89 |
| Random | 280.18 (237.09) | -0.50 | 18120.21 (21660.01) | -1.42 | 95.76 (202.23) | 61.45 | 406.73 (348.30) | -0.82 |
| NV | 259.48 (227.81) | 6.93 | 17295.18 (21860.07) | 3.20 | 245.61 (282.04) | 1.13 | 406.69 (347.72) | -0.81 |
| Eff | 263.60 (229.24) | 5.45 | 16636.52 (21322.89) | 6.88 | 243.30 (276.65) | 2.06 | 417.71 (360.13) | -3.54 |
| SBP | 276.61 (235.84) | 0.78 | 16952.85 (21386.07) | 5.11 | 44.14 (148.60) | 82.23 | 379.23 (326.86) | 5.99 |
| HEM (Ours) | **241.77 (229.97)** | **13.28** | **15751.08 (20683.53)** | **11.84** | **43.88 (148.66)** | **82.34** | **368.51 (309.02)** | **8.65** |

**2** applications, **40%** improvement on average




- Visualize the diversity of selected cuts
- HEM selects much more diverse cuts that can well complement each other nicely

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# Learning Cut Selection for Mixed Integer Programming via Hierarchical Sequence Model

## Summary of this work

- We **observe** from extensive empirical results that the **order** of selected cuts has a significant impact on the efficiency of solving MILPs.
- To the best of our knowledge, our proposed HEM is **the first** method that is able to tackle (P1)-(P3) in cut selection simultaneously from a data-driven perspective.
- We propose to formulate the cut selection task as **a sequence to sequence learning problem**, which not only can capture the underlying order information, but also well captures the interaction among cuts to select cuts that complement each other nicely.
- Extensive experiments demonstrate that HEM **achieves significant improvements** over competitive baselines on challenging MILP problems, including some benchmarks from MIPLIB 2017 and large-scale real-world production planning problems.

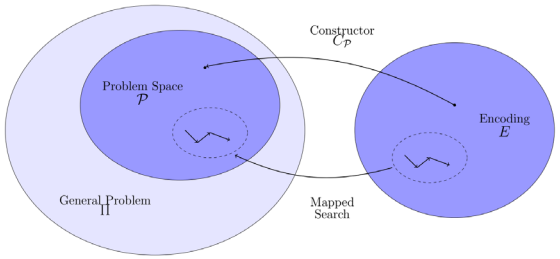
# Challenges and outlook: from learning to game

## Challenge 1

**ML for MIP: Limited Data**

Training Dataset → MIP Solver → Test Dataset

**Generate Synthetic Instances**




Unable to resemble data distributions

## Challenge 2

**Modules in MIP: Too Isolated Training**

| Variable Selection | Node Selection | Cut Selection |
|---|---|---|
| Representation A | Representation B | Representation C |
| Policy A | Policy B | Policy C |

**Unified Representation & Joint Training**

Representation
Policy
→ Module A / Module B / Module C

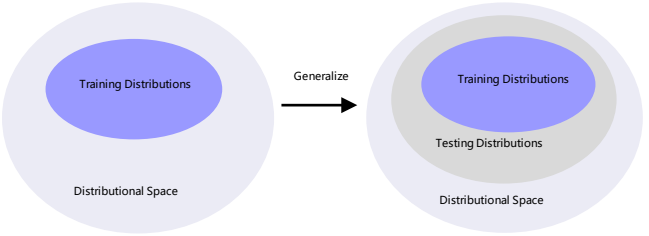
Different modules depend on and affect each other

## Challenge 3

**ML for MIP: Limit Training distributions**

Training Dataset → MIP Solver → Test Dataset

**Improving Generalization on Testing Dataset**




Unable to perform well on a large range of unseen distributions



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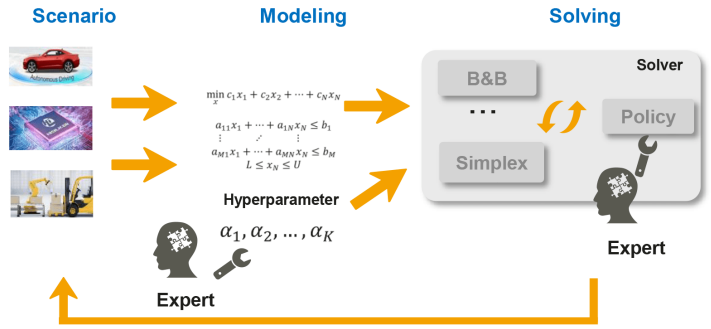
# [Solver] AI Solver with Automatic Algorithmic Reasoning
# 华为黄大年茶思屋挑战难题

https://www.chaspark.com/#/questions/8286690110847877 12?sub=8286690110973706 24




Mathematical programming solver, which is a very important tool for solving optimization problems, is widely used in numerous fields, such as chip design, autonomous driving, resource scheduling, and supply chain optimization.
Solving optimization problems in reality usually consists of two parts: modeling and solving. Optimizing both parts requires the experience of experts and a lot of human effort. Modeling requires a good understanding of the background of the optimization problems — this is a prerequisite for building a model that the business needs. Solving requires a deep understanding of the optimization algorithm and lots of solving policies for different problem structures so that the solver can achieve fast and accurate results in many scenarios.
Generally, two layers of policies will help optimize the solver performance:

- **Outer policy [1]:** Based on the structures of domain-specific problems, experts select a group of optimal parameters out of massive hyperparameters and input them into the solver, or the solver developers set a group of good default parameters out of expert experience.
- **Inner policy:** The solver developers translate expert experience into the following work in order to improve solver stability and performance:
  - ✓ Design an algorithm to discover solver-friendly model structures (such as model rearrangement [3, 4] and massive structure mining).
  - ✓ Design and reason efficient solving policies manually [2] (e.g., branch variable selection and cutting plane selection).

Compared with the outer policy, research on the inner policy is still in its infancy. With the abundant data accumulated in enterprises, we can use AI technologies to implement automatic algorithm reasoning of the solver's inner policy, thus reducing dependency on expert experience and enhancing the solver's competitiveness.

## Technical Challenges

- **Solver-friendly model structure discovery algorithms:**
  - ✓ **Low automation rate:** Structure analysis depends heavily on optimization experts' understanding of the problem and cannot be adaptively implemented by a solver.
  - ✓ **Non-solver-friendly:** Structure discovery algorithms do not perform joint optimization based on solver performance.
- **Automatic design and reasoning of solving policies:**
  - ✓ **High cost of designing new solving policies:** Designing new solving policies requires extensive theoretical knowledge, and once designed, the policies cannot be automatically applied to other problems.
  - ✓ **Joint optimization of multiple solving policies:** Considering the mutual influence between different policies in the solver, the optimal performance can only be achieved through joint optimization.
  - ✓ **Poor generalizability:** It is difficult to obtain a common solving policy by training different types of models.

## Current Scenario & Achievements

- **Manual structure discovery:** In applications such as supply chain production planning and supply network optimization, manually designing structure mining methods based on the production hierarchy or period can improve the solving efficiency by 50%. However, this requires x person-months of design of a structure discovery algorithm, resulting in high costs.
- **Single-module policy learning:** In the current mixed integer programming (MIP) solver, technologies like imitation learning and reinforcement learning are applied in key performance modules (e.g., branch variable selection, cutting plane generation and selection, and pre-solving). Compared with the default solver rules, these technologies have significantly improved the performance of policy learning. When it comes to industrial problems, the average performance improvement even exceeds 20%. However, multi-module joint learning is yet to be optimized.

## Technical Requirements

Design an AI solver that features automatic algorithmic reasoning. Compared with the default policies of the traditional mathematical programming solver (OptVerse AI Solver or SCIP), the solving time on the given dataset (MIPLIB2017 or Huawei-designated) must be reduced by 30%.
MIPLIB 2017: https://miplib.zib.de/tag_benchmark.html


[1] https://www.chaspark.net/#/questions/7167056707658588 16?sub=7167076742045245 46
[2] Fawzi, Alhussein, et al. "Discovering faster matrix multiplication algorithms with reinforcement learning." Nature 610.7930 (2022): 47-53.
[3] Lodi, Andrea, and Andrea Tramontani. "Performance variability in mixed-integer programming." Theory driven by influential applications. INFORMS, 2013. 1-12.
[4] Xijun Li, Qingyu Qu, Fangzhou Zhu, Jia Zeng, Mingxuan Yuan, Jie Wang: Accelerating Linear Programming Solving by Exploiting the Performance Variability via Reinforcement Learning. AAAI 2023 workshop on AI to Accelerate Science and Engineering

# Many thanks & QA

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# Summary of our group’s work in last two years

[1] Xijun Li*, Zhihai Wang*, Jie Wang, Yufei Kuang, Mingxuan Yuan, Jia Zeng, Yongdong Zhang, Feng Wu: Learning Cut Selection for Mixed-Integer Linear Programming via Hierarchical Sequence Model. ICLR 2023 [pdf]

[2] Han Lu, Zenan Li, Runzhong Wang, Qibing Ren, Xijun Li, Mingxuan Yuan, Jia Zeng, Xiaokang Yang, Junchi Yan: ROCO: A General Framework for Evaluating Robustness of Combinatorial Optimization Solvers on Graphs. ICLR 2023 [pdf]

[3] Xijun Li, Qingyu Qu, Fangzhou Zhu, Jia Zeng, Mingxuan Yuan, Jie Wang: Accelerating Linear Programming Solving by Exploiting the Performance Variability via Reinforcement Learning. AAAI 2023 workshop on AI to Accelerate Science and Engineering

[4] Wenxuan Guo, Hui-Ling Zhen, Xijun Li#, Mingxuan Yuan, Yaohui Jin, Junchi Yan: Machine Learning Methods in Solving the Boolean Satisfiability Problem. Machine Intelligence Research Journal

[5] Jiayi Zhang, Chang Liu, Xijun Li#, Hui-Ling Zhen, Junchi Yan, Mingxuan Yuan: A Survey for Solving Mixed Integer Programming via Machine Learning. Neurocomputing Journal [pdf]

[6] Ji Zhang, Xijun Li#, Xiyao Zhou, Mingxuan Yuan, Zhuo Cheng, Keji Huang, Yifan Li: L-QoCo: learning to optimize cache capacity overloading in storage systems. DAC 2022: 379-384 (CORE A, CCF A) [pdf]

[7] Xijun Li*#, Yunfan Zhou*, Jinhong Luo, Mingxuan Yuan, Jia Zeng, Jianguo Yao: Learning to Optimize DAG Scheduling in Heterogeneous Environment. MDM 2022: 137-146 (CORE A, CCF C) [pdf] [video] [slides]

[8] Qingyu Qu, Xijun Li#, Yunfan Zhou; Yordle: An Efficient Imitation Learning for Branch and Bound; NeurIPS 2021 ML4CO Competition [pdf]

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